Scaling limits of directed polymers in spatial-correlated environment

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The 16th Workshop on Markov Processes and Related Topics

CSU and BNU, July 12-16, 2021

Joint work with Yingxia Chen

Outline

Directed polymer

- Directed polymer: a random probability distribution on the path space $(\mathbb{Z}^d)^{\mathbb{Z}_+}$

$$\mathbb{P}^{\omega}_{n,\beta}(S) := \frac{1}{Z_n(\beta,\omega)} e^{\beta \sum_{i=1}^{n} \omega(i,S_i)} \mathbb{P}(S), \tag{1.1}$$

  - where $\beta > 0$ is the inverse temperature,
  - $S = \{S_n, n \geq 0\}$ is a random walk in $\mathbb{Z}^d$ on $((\mathbb{Z}^d)^{\mathbb{Z}_+}, \mathcal{F}^{\mathcal{S}}, \mathbb{P})$,
  - $\omega = \{\omega(i,x), (i,x) \in \mathbb{Z}_+ \times \mathbb{Z}^d\}$ is an environment which is a family of identically distributed random variables on $(\Omega, \mathcal{F}^{\Omega}, \mathbf{P})$,
  - $Z_n(\beta,\omega)$ is the point-to-line partition function defined by

$$Z_n(\beta;\omega) := \mathbb{E}\left(e^{\beta \sum_{i=1}^{n} \omega(i,S_i)}\right). \tag{1.2}$$

- D. A. Huse, C. L. Henley. Pinning and roughening of domain walls in Ising systems due to random impurities. *Phys. Rev. Lett.* 54(1985), 2708-2711.
- J. Z. Imbrie, T. Spencer. Diffusion of directed polymers in a random environment. *J. Stat. Phys.* 52(1988), 609-626.
- F. Comets. *Directed polymers in random environments*, *Lecture Notes in Mathematics 2175*. Springer, 2017

- The free energy:

$$p_n(\beta) := \frac{1}{n} \log Z_n(\beta; \omega)$$

- The point-to-point partition function

$$Z_{n,x}(\beta; \omega) := \mathbb{E}\left(e^{\beta \sum_{i=1}^{n} \omega(i,S_i)} I_{\{S_n = x\}}\right). \tag{1.3}$$

- The polymer endpoint distribution

$$\mathbb{P}^{\omega}_{n,\beta}(S_n = x) := \frac{Z_{n,x}(\beta; \omega)}{Z_n(\beta, \omega)}. \tag{1.4}$$

- To study the behaviors of the the polymer as $n \to \infty$, and as d and β vary.
- Fluctuation exponents for the polymer endpoint and the free energy:

$$E_{\mathbb{P}^{\omega}_{n,\beta}}(S_n) \sim n^{2\zeta}, \quad \mathrm{Var}_{\mathbb{P}^{\omega}_{n,\beta}}(\log Z_n(\beta;\omega)) \sim n^{2\chi},$$

- **P**-*a.s.*

$$p(\beta) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta;\omega).$$

- At $\beta = 0$, the polymer measure is the simple random walk, the polymer exhibits diffusive behavior. Weak disorder
- For β large, the polymer measure concentrates on paths with high energy. Strong disorder

- Assume that for β sufficiently small,

$$\lambda(\beta) := \log \mathbf{E} e^{\beta\omega(i,x)} < \infty. \tag{1.5}$$

Then

$$p(\beta) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta;\omega) = \lim_{n\to\infty} \frac{1}{n} \log \mathbf{E}\left(\log Z_n(\beta;\omega)\right) < \lambda(\beta).$$

- The normalized partition function

$$W_n := Z_n(\beta;\omega) \exp\{-n\lambda(\beta)\}, \; n \geq 1. \tag{1.6}$$

- **P**-*a.s.*

$$W_\infty = \lim_{n\to\infty} W_n \tag{1.7}$$

exists and either the polymer is

$$\begin{cases} \text{in weak disorder regime,} & i.e., \mathbf{P}(W_\infty > 0) = 1; \\ \text{or in strong disorder regime,} & i.e., \mathbf{P}(W_\infty = 0) = 1 \end{cases}$$

- When $d = 1$, all $\beta > 0$ are in the strong disorder regime.

The intermediate disorder regime

- Alberts, Khanin and Quastel (2014) introduced a new disorder regime: the intermediate disorder regime.
  - $d = 1$, the environment i.i.d., $\{S_n\}$ simple symmetric random walk.
  - The scaled partition function:

$$Z_n(n^{-1/4}\beta;\omega)e^{-n\lambda(n^{-1/4}\beta)} \xrightarrow{(d)} \mathcal{Z}_{\sqrt{2}\beta}$$

  - The scaled point-to-point partition function

$$\frac{1}{2}\sqrt{n}Z_{nt,\sqrt{n}x}(n^{-1/4}\beta;\omega)e^{-n\lambda(n^{-1/4}\beta)} \xrightarrow{(d)} \mathcal{Z}_{\sqrt{2}\beta}(t,x) \quad \text{in } C([0,1]\times\mathbb{R}),$$

where $\mathcal{Z}_{\sqrt{2}\beta} = \int \mathcal{Z}_{\sqrt{2}\beta}(1,x)dx$ and $u(t,x) := \mathcal{Z}_{\sqrt{2}\beta}(t,x)$ is the mild solution of the stochastic heat equation

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u + \sqrt{2}\beta u\dot{W}, \\ u(0,x) = \delta_x. \end{cases} \tag{1.8}$$

- T. Alberts, K. Khanin, and J. Quastel. Intermediate disorder regime for directed polymers in dimension $1+1$. *Phys. Rev. Lett.*, 105(9)(2010),090603.
- T. Alberts, K. Khanin, and J. Quastel. The intermediate disorder regime for directed polymers in dimension $1+1$. *Ann. Probab.* 42(2014), 1212–1256.

- The polymer transition probabilities

$$\left\{(s,y;t,x) \mapsto \frac{\sqrt{n}}{2}\mathbf{P}^{\omega}_{n,\beta_n}(S_{nt} = x\sqrt{n} | S_{ns} = y\sqrt{n})\right\}$$

$$\xrightarrow{(d)} \frac{\mathcal{Z}_{\sqrt{2}\beta}(s,y;t,x)\int \mathcal{Z}_{\sqrt{2}\beta}(t,x;1,\lambda)\,d\lambda}{\mathcal{Z}_{\sqrt{2}\beta}}$$

  for $0 \le s < t \le 1$ and $x, y \in \mathbb{R}$.
- $\mathcal{Z}_\beta(s,y;t,x)$ is the mild solution of the stochastic heat equation

$$\partial_t \mathcal{Z}_\beta = \tfrac{1}{2}\partial_{xx}\mathcal{Z}_\beta + \beta \mathcal{Z}_\beta \dot{W}, \qquad \mathcal{Z}_\beta(s,y;s,x) = \delta_0(x-y),$$

- Under the scaling $\beta_n = \beta n^{-(1/4+\delta)}$ for any $\delta > 0$ (Supercritical scaling):
    - the partition function $e^{-n\lambda(\beta_n)} Z_n^\omega(\beta n^{-(1/4+\delta)})$ converges in probability to 1;
    - the endpoint density, under diffusive scaling of space, converges to the standard Gaussian distribution.
- The scalings $\beta_n := \beta n^{-\alpha}$ for $0 \le \alpha < 1/4$ (Subcritical scaling):
    - The individual terms of the discrete Wiener chaos blow up as $n \to \infty$.

KPZ and Scaling limits

- Since the logarithm of solution of the stochastic heat equation is the Cole-Hopf solution of Kardar-Parisi-Zhang (KPZ) equation:

$$\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}(\nabla h)^2 + \sqrt{2}\beta \dot{W}, \tag{1.9}$$

- Alberts, Khanin and Quastel (2014) have really derived the KPZ equation from the scaling limit of the directed polymer.
- These results have been extended to many new models.

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- J. Quastel, *Introduction to KPZ*. Current developments in mathematics, 2011, Int. Press, Somerville, MA, 2012, 125–194.

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- Caravenna, Sun and Zygouras (JEMS, 2017) provided a unified framework to study the scaling limits of some statistical mechanics systems.
- Joseph (SPDEAC, 2018) considered an appropriate scaling limit of a model of discrete space-time stochastic heat equations.

$$\partial_t u = -\nu_\alpha(-\Delta)^{\alpha/2} u + \sigma(u)\dot{W}, \qquad (1.10)$$

  where σ is Lipschitz continuous.
- Rang (SPA, 2020) considered time independent and space correlated environment.
- Furthermore, see Corwin, Nica (EJP, 2017), Clement (Ind. Math, 2019), Shen et. all. (2000) and the references therein.

Model and assumptions

- We consider the directed polymer involving random walks attracted to stable laws, and time-independent and space-correlated environment.

$$\mathbb{P}^{\omega}_{n,\beta}(S) := \frac{1}{Z_n(\beta,\omega)} e^{\beta \sum_{i=1}^{n} \omega(i,S_i)} \mathbb{P}(S),$$

- **(A.1).** Let the random walk $\{S_n, n \geq 0\}$ be in the domain of attraction of a stable law of index $\alpha \in (1,2]$ with period q Define

$$p(n,k) := \mathbb{P}(S_n = k), \qquad n \geq 0,\ k \in \mathbb{Z},$$

and

$$p(nt,kx) := p([nt],[kx]), \qquad n \geq 0,\ k \in \mathbb{Z},\ t \in [0,1],\ x \in \mathbb{R}$$

- Let $g(x)$ be the density of symmetric α-stable distribution.

$$g(t,x) := \frac{1}{t^{1/\alpha}} g\left(\frac{x}{t^{1/\alpha}}\right), \quad t > 0,\ x \in \mathbb{R}.$$

- **(A.2).** The environment $\omega = \{\omega(i,x), (i,x) \in \mathbb{Z}_+ \times \mathbb{Z}^d\}$:

$$\omega(i,x) = \sum_{-\infty<y<+\infty} a_y \xi(i, x+y), \qquad a_y \sim \delta |y|^{-r},$$

  where $1/2 < r < 1, \delta > 0, \{\xi(i,x) : i \in \mathbb{Z}_+, x \in \mathbb{Z}\}$ is a family of independent identical distribution variables with $\mathbf{E}(\xi(i,x)) = 0, \mathbf{E}(|\xi(i,x)|^2) = 1$.

$$\mathbf{E} e^{\beta|\xi(i,x)|} < \infty \tag{1.11}$$

  for β sufficiently small which implies (1.5).

$$\mathbf{E}(\omega(i,x)\omega(j,y)) = \delta_{ij}\gamma(x-y),$$

  where $\gamma(z) \sim \frac{1}{2q}\left(|z-q|^{3-2r} + |z+q|^{3-2r} - 2|z|^{3-2r}\right)$ as $|z| \to \infty$.

- G. L. Rang. From directed polymers in spatial-correlated environment to stochastic heat equations driven by fractional noise in 1+1 dimensions. *Stoch. Proce. Appl.* 130(2020), 3408-3444.

Multiple stochastic integral

- Let $K(x) = H(2H-1)|x|^{2H-2}$, $H = \frac{3}{2} - r$.
- A time-white spatial-colored noise with the kernel K: a mean zero Gaussian process $\{\mathcal{W}(\phi), \phi \in \mathcal{S}([0,1]\times\mathbb{R})\}$,

$$\mathrm{Cov}(\mathcal{W}(\phi), \mathcal{W}(\psi)) = \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(s,x) K(x-y) \psi(s,y) \mathrm{d}s \mathrm{d}x \mathrm{d}y.$$

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$$\mathcal{L}_H^k = \Big\{ f : ([0,1]\times\mathbb{R})^k \to \mathbb{R};$$

$$\| f \|_{\mathcal{L}_H^k}^2 := \int_{\Delta_k(0,1]} \int_{\mathbb{R}^{2k}} f(\mathbf{t},\mathbf{x}) \prod_{i=1}^k K(x_i - y_i) f(\mathbf{t},\mathbf{y}) \mathrm{d}\mathbf{t} \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{y} < \infty$$

where $\mathbf{t} = (t_1, t_2, \cdots, t_k)$, $\mathbf{x} = (x_1, x_2, \cdots, x_k)$, $\mathbf{y} = (y_1, y_2, \cdots, y_k)$, and

$$\Delta_k(0,t] = \{0 = t_0 < t_1 < t_2 < \cdots < t_k < t\}.$$

- For $f \in \mathcal{L}_H$, the stochastic integral $\mathcal{W}(f)$ with respect to $\mathcal{W}$ is defined by

$$I_1^{\mathcal{W}}(f) := \mathcal{W}(f) := \sum_{n \geq 1} \langle f, h_n \rangle_{\mathcal{L}_H} \mathcal{W}(h_n).$$

- $$I_k^{\mathcal{W}}(f^{\otimes k}) := \int_{([0,1]\times\mathbb{R})^k} f^{\otimes k}(\mathbf{t}, \mathbf{x}) \mathcal{W}^{\otimes k}(\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{x}) := \mathrm{H}_k(\mathcal{W}(f)),$$

- $f \in \mathcal{L}_H^k$,

$$I_k^{\mathcal{W}}(f) := \int_{([0,1]\times\mathbb{R})^k} f(\mathbf{t}, \mathbf{x}) \mathcal{W}^{\otimes k}(\mathrm{d}\mathbf{t}\mathrm{d}\mathbf{x}).$$

- $$\mathrm{Cov}(I_j^{\mathcal{W}}(f), I_k^{\mathcal{W}}(g)) = \begin{cases} k! \langle f, g \rangle_{\mathcal{L}_H^k} & \text{if } j = k, \quad f, g \in \mathcal{L}_H^k \\ 0 & \text{if } j \neq k. \end{cases}$$

Stochastic heat equation

- Consider the following stochastic heat equation:

$$\partial_t u = -\nu_\alpha(-\Delta)^{\alpha/2}u + \sqrt{q}\beta u\dot{\mathcal{W}}, \tag{1.12}$$

- The mild solution solution with initial data $u_0 = u(0,x)$ can be written by

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}} g(t,x-y)u(0,y)\mathrm{d}y \\ &+ \sum_{k=1}^{\infty}(\sqrt{q}\beta)^k \int_{\Delta_k(0,t]}\int_{\mathbb{R}^k} g(t-t_k, x-x_k) \\ &\prod_{i=1}^{k} g(t_i - t_{i-1}, x_i - x_{i-1})\mathcal{W}(\mathrm{d}t_i\mathrm{d}x_i), \end{aligned} \tag{1.13}$$

where $t_0 = 0, x_0 = x$.

Main results

Theorem 2.1

Assume that (A.1) and (A.2) hold. Set $\beta_n = \beta n^{-\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}}$. *Then the scaled point-to-line partition*

$$Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)} \xrightarrow{(d)} u(1,0), \tag{2.1}$$

and

$$\lim_{n\to\infty} \mathbf{E}\left(\left(Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)}\right)^2\right) = \mathbf{E}\left((u(1,0))^2\right),$$

where $u(t,x)$ *is the mild solution of (1.12) with initial data* $u_0 = 1$.

Theorem 2.2

Let $\frac{1}{2} < r < \min\{1, \alpha - \frac{1}{2}\}$*. Assume that (A.1) and (A.2) hold. Then the scaled point-to-point partition*

$$\frac{1}{q} n^{1/\alpha} Z_{nt, n^{1/\alpha} x}(\beta_n; \omega) e^{-n\lambda(\beta_n)} \xrightarrow{(d)} u(t, x), \tag{2.2}$$

in the sense of the finite dimensional distributions in $C([0, 1] \times \mathbb{R})$*, and*

$$\lim_{n \to \infty} \mathbf{E} \left(\left(n^{1/\alpha} Z_{nt, n^{1/\alpha} x}(\beta_n; \omega) e^{-n\lambda(\beta_n)} / q \right)^2 \right) = \mathbf{E} \left((u(t, x))^2 \right),$$

where $u(t, x)$ *is the mild solution of (1.12) with initial data* $u_0(x) = \delta(x)$*.*

- Furthermore, if

$$\phi(u) := \mathbb{E}\left(e^{\sqrt{-1}uS_1}\right) = 1 - \nu_\alpha |u|^\alpha + h(u), \tag{2.3}$$

where $h(u) = o(|u|^\alpha)$ as $|u| \to 0$, then

$$\frac{1}{q} n^{1/\alpha} Z_{nt, n^{1/\alpha}x}(\beta_n; \omega) e^{-n\lambda(\beta_n)} \xrightarrow{(d)} u(t,x), \quad \text{in } C([0,1] \times \mathbb{R}), \tag{2.4}$$

where the topology is the supremum norm.

- The polymer transition probabilities

$$\left\{(s,y;t,x)\mapsto \frac{1}{q}n^{1/\alpha}\mathbf{P}^{\omega}_{n,\beta_n}(S_{nt}=n^{1/\alpha}x|S_{ns}=n^{1/\alpha}y)\right\}$$
$$\xrightarrow{(d)} \frac{\mathcal{Z}_{\sqrt{q}\beta}(s,y;t,x)\int \mathcal{Z}_{\sqrt{q}\beta}(t,x;1,\lambda)\,d\lambda}{\mathcal{Z}_{\sqrt{q}\beta}}$$

  for $0\le s<t\le 1$ and $x,y\in\mathbb{R}$.
- $\mathcal{Z}_\beta(s,y;t,x)$ is the mild solution of the stochastic heat equation

$$\partial_t\mathcal{Z}_\beta=-\nu_\alpha(-\Delta_x))^{\alpha/2}\mathcal{Z}_\beta+\beta\mathcal{Z}_\beta\dot{W},\ \ \mathcal{Z}_\beta(s,y;s,x)=\delta_0(x-y),$$

- Foondun Joseph and Li (AAP, 2018) studied the approximation problem of a class of SPDEs, including (1.12), by systems of interacting stochastic differential equations. Our results show that the solution $u(t,x)$ of (1.12) is the limit of the scaled point-to-point partition function of a directed polymer.
  - M. Foondun, M. Joseph, S. T. Li. An approximation result for a class of stochastic heat equations with colored noise. *The Annals of Applied Probability*. 28(2018), 2855–2895.

Proof of Theorem 2.1

- Consider the modified point-to-line partition function:

$$\mathfrak{Z}_n(\beta;\omega) = \mathbb{E}\left(\prod_{i=1}^{n}(1+\beta\omega(i,S_i))\right), \tag{3.1}$$

- Then

$$\mathfrak{Z}_n(\beta_n;\omega) = 1 + \sum_{k=1}^{n} \beta_n^k p_n^k(\mathbf{t},\mathbf{x}) \left(\prod_{i=1}^{k} \omega\left(nt_i, n^{\frac{1}{\alpha}} x_i\right)\right).$$

where

$$p_n^k(\mathbf{t},\mathbf{x}) := \prod_{i=1}^{k} p(n(t_i - t_{i-1}), n^{\frac{1}{\alpha}}(x_i - x_{i-1})), \quad (\mathbf{t},\mathbf{x}) \in \Delta\mathbb{D}_n^k,$$

$$\Delta\mathbb{D}_n^k := \left\{(\mathbf{t},\mathbf{x}) = ((t_1,x_1),\cdots,(t_k,x_k)) \in \mathbb{D}_n^k : 0 \le t_1 < \cdots < t_k \le 1\right\}$$

$$\mathbb{D}_n := \left\{\left(\frac{i}{n}, \frac{x}{n^{\frac{1}{\alpha}}}\right) : x \in q\mathbb{Z} + il, 1 \le i \le n\right\}$$

- Define

$$\mu(i,x) = \sum_{-\infty}^{+\infty} a_y \eta(i, x+y),$$

where $\{\eta(i,x),(i,x) \in \mathbb{Z}_+ \times \mathbb{R}\}$ is a family of i.i.d. standard Gaussian random variables, and independent of $\{\xi(i,x),(i,x) \in \mathbb{Z}_+ \times \mathbb{R}\}$.

- Define

$$\mathfrak{Z}_n(\beta_n;\mu) = 1 + \sum_{k=1}^{n} \beta_n^k g_k(\mathbf{t},\mathbf{x}) \left(\prod_{i=1}^{k} \mu\left(nt_i, n^{\frac{1}{\alpha}} x_i\right)\right).$$

where

$$g_k(\mathbf{t},\mathbf{x}) := \prod_{i=1}^{k} g(t_i - t_{i-1}, x_i - x_{i-1}) \qquad (\mathbf{t},\mathbf{x}) \in \Delta_k(0,1] \times \mathbb{R}^k.$$

- $$\mathfrak{Z}_n(\beta_n;\omega) \xrightarrow{(d)} 1+\sum_{k=1}^{\infty}(\beta\sqrt{q})^k \int_{\Delta_k(0,1]}\int_{\mathbb{R}^k} g_k(\mathbf{s},\mathbf{y}) \prod_{i=1}^{k} \mathcal{W}(\mathrm{d}s_i \mathrm{d}y_i).$$

- $$\lim_{n\to\infty} \mathbf{E}\left(\left(Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)}\right) - \mathfrak{Z}_n(\beta_n;\omega)\right)^2 = 0.$$

- Therefore

$$Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)} \xrightarrow{(d)} 1+\sum_{k=1}^{\infty}(\beta\sqrt{q})^k \int_{\Delta_k(0,1]}\int_{\mathbb{R}^k} g_k(\mathbf{s},\mathbf{y}) \prod_{i=1}^{k} \mathcal{W}(\mathrm{d}s_i \mathrm{d}y_i).$$

Proof of Theorem 2.2

- Consider the modified point-to-point partition defined by

$$\mathfrak{Z}_{n,x}(\beta;\omega) = \mathbb{E}\left(\prod_{i=1}^{n}\left(1+\beta\omega(i,S_i)\right) I_{\{S_n=x\}}\right).$$

- Then

$$\begin{aligned}&\mathfrak{Z}_{n,n^{1/\alpha}x}(\beta_n;\omega)\\ =&p(n,n^{1/\alpha}x)\left(1+\sum_{k=1}^{n}\beta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\Delta\mathbb{D}_n^k}\psi_{n,x}^k(\mathbf{t},\mathbf{x})\prod_{i=1}^{k}\omega(nt_i,n^{1/\alpha}x_i)\right)\end{aligned}$$

where

$$\psi_{n,x}^k(\mathbf{t},\mathbf{x}) = \frac{p(n(1-t_k),n^{1/\alpha}(x-x_k))}{p(n,n^{1/\alpha}x)}\prod_{i=1}^{k}p(n(t_i-t_{i-1}),n^{1/\alpha}(x_i-x_{i-1})).$$

- Denote by

$$z_n(t,x) := n^{\frac{1}{\alpha}} \mathfrak{Z}_{nt,n^{1/\alpha}x}(\beta_n;\omega), \quad \bar{z}_n(t,x) = \mathcal{P} z_n(t,x)$$

  where $\mathcal{P}$ is the transition probability of $\{S_n\}$.
- Then

$$z_n(t,x) = p_n(t,x) + \beta \int_0^t \int_{\mathbb{R}} p_n(t-s,y-x)\bar{z}_n(s,y)\omega_n(s,y)\mathrm{d}s\mathrm{d}y. \tag{3.2}$$

  where

$$p_n(t,x) = n^{\frac{1}{\alpha}} p([nt],[n^{\frac{1}{\alpha}}x])$$

$$\omega_n(s,y) = n^{\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}} \omega([ns],[n^{\frac{1}{\alpha}}y]).$$

- Convergence of finite dimensional distributions: Proof of Theorem 2.1
- Tightness:
  - $$\sup_{t\in[\varepsilon,1],x\in\mathbb{R}} \mathbf{E}\left(z_n^{2m}(t,x)\right) \le C_m, \ \sup_{t\in[\varepsilon,1]} \int_{\mathbb{R}} \mathbf{E}\left(z_n^{2m}(t,x)\right) \mathrm{d}x \le C_m$$
  - For all $t \ge \varepsilon$,

$$\begin{aligned} &\mathbf{E}\left(z_n(t+h,x+\delta) - z_n(t,x)\right)^{2m} \\ \le &C_m \left(h^{\{(1-\frac{1}{\alpha})(2r-1)m\}\wedge\{\frac{m}{\alpha}\}} + \delta^{\{(1-\frac{1}{\alpha})(2r-1)m\}\wedge\{\frac{m}{\alpha}\}}\right). \end{aligned}$$

    Choose m large enough such that $\{(1-\frac{1}{\alpha})(2r-1)m\}\wedge\{\frac{m}{\alpha}\} > 2$.

T. Alberts, K. Khanin, and J. Quastel. The intermediate disorder regime for directed polymers in dimension $1+1$. *Ann. Probab.* 42(2014), 1212–1256.

G. Amir, I. Corwin, and J. Quastel. Probability distribution of the free energy of the continuum directed random polymer in $1+1$ dimensions. *Comm. Pure Appl. Math.* 64 (2011), 466–537.

F. Caravenna, R. Sun, N. Zygouras. Polynomial chaos and scaling limits of disordered systems. *J. Eur. Math. Soc.*19(2017),1–65.

E. Mossel, R. O'Donnell and K. Oleszkiewicz. Noise stability of functions with low influences: Invariance and optimality. *Ann. Math.*171(2010), 295–341.

G. L. Rang. From directed polymers in spatial-correlated environment to stochastic heat equations driven by fractional noise in 1+1 dimensions. *Stochastic Process. Appl.* 130(2020), 3408-3444.

Thank you!